The origin of scale-scale correlations of the density perturbations during inflation

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We show that scale-scale correlations are a generic feature of slow-roll inflation theories. These correlations result from the long-time tails characteristic of the time dependent correlations because the long wavelength density perturbation modes are diffusion-like. A relationship between the scale-scale correlations and time-correlations is established providing a way to reveal the time correlations of the perturbations during inflation. This mechanism provides for a testable prediction that the scale-scale correlations at two different spatial points will vanish.

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The inflation paradigm for the evolution of the universe not only solves classical cosmological dilemmas such as the horizon and flatness problems, but also provides the seeds for structure formation in the form of quantum or thermal fluctuations [1]. These fluctuations grow by gravitational instability to form the structure we see today. The nature of the fluctuations leads to the belief that the primordial density field is likely Gaussian. However, recent analysis of Lyα forest lines in quasar absorption spectra reveal that the Lyα clouds at redshifts between 2 and 3 are significantly scale-scale correlated on scales of 40 to 80 h^{-1} Mpc [2]. Since the gravitational clustering of the universe remains in the linear regime on scales larger than about 30 h^{-1} Mpc at redshifts higher than 2, this result implies that the primordial density fluctuations of the universe are probably scale-scale correlated. More direct evidence for the scale-scale correlation in the primordial density field has recently been found in the Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) 4-year all sky maps [3]. In particular, the observed cosmic temperature fluctuations on the North Galactic Pole are found, with $> 99\%$ confidence, to be scale-scale correlated on angular scales of 10 to 20 degrees. These scales are much larger than the Hubble size at the decoupling era and cannot be attributed to any known foregrounds or correlated noise maps. Moreover, the correlation is on the order of $\simeq 10^{-5}$ K, i.e., the same order as the temperature fluctuations, and therefore it is unlikely to be a higher order effect.

Although confirmation of scale-scale correlations in the primordial density field must await further observations, it is appropriate now to study the possible origin of these correlations. At the first glance, these correlations might appear to be incompatible with inflation. However, in this *Letter*, we show that scale-scale correlations are not only possible, but a generic feature of the inflationary scenario.

It is known that there are models which lead to a density field with a Poisson or Gaussian distribution in the one-point distribution functions, but that are highly scale-scale correlated [4]. The physical reason for the co-existence of Gaussian one-point functions and scale-scale correlations can easily be seen in phase space, i.e., position-wavevector space. Any density field $\rho(\mathbf{r})$ can be decomposed by an orthogonal and complete basis $\Psi_{\mathbf{k},\mathbf{x}}$, where the indexes $\mathbf{k}$ and $\mathbf{x}$ denote, respectively, the wavevector and position of a volume element $d^3x d^3k \simeq 1$ in phase space. One way to realize this decomposition is by the discrete wavelet transform (DWT) [5]. With this type of decomposition, the behavior of $\rho(\mathbf{r})$ in phase space is described by the projection of $\rho(\mathbf{r})$ on $\Psi_{\mathbf{k},\mathbf{x}}$, which we denote by $\tilde{\epsilon}_{\mathbf{k},\mathbf{x}}$. The density perturbations of component $\tilde{\epsilon}_{\mathbf{k},\mathbf{x}}$ are localized at $\mathbf{k},\mathbf{x}$ in phase space, and thus one can have distributions for which $\tilde{\epsilon}_{\mathbf{k},\mathbf{x}}$ is Gaussian in its one-point distribution with respect to $\mathbf{x}$ while scale-scale correlated in terms of $\mathbf{k}$. For instance, consider a distribution of $\tilde{\epsilon}_{\mathbf{k},\mathbf{x}}$ which is Gaussian in $\mathbf{x}$ at a given scale $|\mathbf{k}| = k_1$. Then

$$\langle \tilde{\epsilon}_{\mathbf{k},\mathbf{x}} \tilde{\epsilon}_{\mathbf{k},\mathbf{x}'} \rangle = P_{dwt}(k_1)\delta_{\mathbf{x},\mathbf{x}'}, \tag{1}$$

where $P_{dwt}(k_1)$ is the variance or the power spectrum with respect to the DWT decomposition [6]. All higher order correlations of $\tilde{\epsilon}_{\mathbf{k},\mathbf{x}}$ are zero at this scale. If perturbation $\tilde{\epsilon}_{\mathbf{k},\mathbf{x}}$ at scale $|\mathbf{k}| = k_2$ is related to that at scale $|\mathbf{k}| = k_1$ as

$$\tilde{\epsilon}_{|\mathbf{k}|=k_2,\mathbf{x}} = \alpha \tilde{\epsilon}_{|\mathbf{k}|=k_1,\mathbf{x}}, \tag{2}$$

where α is a constant, the distribution of $\tilde{\epsilon}_{|\mathbf{k}|=k_2,\mathbf{x}}$ is also Gaussian in $\mathbf{x}$ with variance $P_{dwt}(k_2) = \alpha^2 P_{dwt}(k_1)$, but is strongly scale-scale correlated. That is, at a given position $\mathbf{x}$, the scale k_2 perturbation is always proportional to the scale k_1 perturbation. Scale-scale correlations depend only on the statistical behavior of the fluctuation distribution along the $\mathbf{k}$ direction but localized in the $\mathbf{x}$ direction of the phase space. Obviously such scale-scale correlations can only be effectively detected by a scale-space decomposition.

In the inflationary scenario, the statistical behavior of the primeval density perturbations in the $\mathbf{k}$ direction are determined by the time-dependent correlations of the fluctuations during inflation. This is because the density perturbations on comoving wavenumber k originate from fluctuations crossing over the Hubble radius at time t_c, and

$$k \propto e^{Ht_c}, \tag{3}$$

where H is the Hubble constant during inflation. Equation (3) is a well known feature of inflation: the larger the scale of the perturbations, the earlier the time of the formation [1]. Equation (3) is actually a mapping of time t to scale space k of the density perturbations. Thus, the time-time correlations during the inflation, $\langle \phi(t)\phi(0) \rangle$, where $\phi(t)$ denotes fluctuations of the inflaton Φ, will be mapped into scale-scale correlations.

The primeval perturbations will not be scale-scale correlated if the time dependent correlation functions decay faster than Hubble time H^{-1}. In general, the time-time correlation functions of fluctuations in equilibrium decay exponentially within the mean-free time t_f between collisions, which is not longer than about $1/H$. Such fluctuations do not lead to scale-scale correlations. However, it is well known both experimentally and theoretically that there is a generic mechanism for long-range correlations in *nonequilibrium* systems. For these systems, the temporal correlation functions of perturbations decay only as a power-law,

$$\langle \phi(t)\phi(0) \rangle \propto (t_f/t)^{d/2}, \tag{4}$$

where d is the dimension of the system. This long-time tail phenomenon is found to be applicable to all systems where the relaxation is dominated by long wavelength perturbations via diffusion-like dissipation [7,8]. Inflation can be treated as such a system thus implying the existence of scale-scale correlations.

We can confirm this by considering a standard slow-roll inflation governed by a scalar field Φ satisfying the equation of motion as

$$\ddot{\Phi} + 3H\dot{\Phi} - e^{-2Ht}\nabla^2\Phi + V'(\Phi) = 0, \tag{5}$$

where $V(\Phi)$ is the potential of Φ. For our purposes, it is convenient to employ the stochastic inflation formalism [9,10], in which the Φ field is decomposed into large scale component $\phi(\mathbf{x}, t)$ and small scale modes $q(\mathbf{x}, t)$ as

$$\Phi(\mathbf{x}, t) = \phi(\mathbf{x}, t) + q(\mathbf{x}, t). \tag{6}$$

The component $\phi(\mathbf{x}, t)$ is actually a coarse-grained Φ over size $1/k_c > 1/H$. It contains the uniform background and perturbations on scales larger than $1/k_c$, and therefore can be treated classically. (Here and below we use k_c and k to describe the wavenumbers of the fluctuations during inflation. It should not be confused with the k in Eq. (3), where it is the wavenumber of the density perturbations after inflation). The term $q(\mathbf{x}, t)$ contains all high frequency quantum or thermal fluctuations. For slow-roll inflation, the evolution of $\phi(\mathbf{x}, t)$ modes obeys a Langevin equation

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = -\frac{1}{3H}\frac{\delta F[\phi(\mathbf{x})]}{\delta \phi} + \eta(\mathbf{x}, t), \tag{7}$$

where $F[\phi] = \int d^3\mathbf{x} \left[(e^{-Ht}\nabla\phi)^2/2 + V(\phi) \right]$ is the Ginzburg-Landau free energy. The noise term is given by the short-wavelength modes

$$\eta(\mathbf{x}, t) = \left(-\frac{\partial}{\partial t} + \frac{1}{3H}e^{-2Ht}\nabla^2 \right) q(\mathbf{x}, t), \tag{8}$$

and its time correlation function is approximately

$$\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle \simeq D\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'), \tag{9}$$

where $D = H^3/2\pi$ or $H^2 T/2\pi$ (T being temperature of heat bath) for quantum or thermal fluctuations, respectively. As expected, Eq. (9) shows that the short wavelength modes rapidly decay and are Gaussian.

However, the primordial density perturbations originate from the long wavelength modes ϕ. The time correlation behavior of ϕ can be seen from a mode analysis of Eq. (7). Linearizing Eq. (7), one has

$$\frac{\partial \phi}{\partial t} = -\left[\frac{\phi}{\tau_0} - \frac{e^{-2Ht}}{3H}\nabla^2\phi \right] + \eta \tag{10}$$

with the zero wavenumber relaxation time given by

$$\tau_0^{-1} = V''(0)/3H. \tag{11}$$

Taking the Fourier transform of Eq. (10) gives

2

$$\frac{\partial \hat{\phi}_{\mathbf{k}}}{\partial t} = -\frac{\hat{\phi}_{\mathbf{k}}}{\tau_{\mathbf{k}}} + \eta_{\mathbf{k}}, \tag{12}$$

where $\mathbf{k}$ is a comoving wave-vector. The relaxation time of the mode $\mathbf{k}$ is

$$\frac{1}{\tau_{\mathbf{k}}} = \frac{1}{\tau_0} + \frac{1}{3H}e^{-2Ht}k^2 = \frac{1}{\tau_0} + \frac{k_p^2(t)}{3H}, \tag{13}$$

where $k_p(t) = k\exp(-Ht)$ is the physical wavenumber of k. Equation (13) is a dispersion relation of mode $\mathbf{k}$, i.e., $i\omega = \tau_{\mathbf{k}}^{-1}$. If the potential V is flat, $V' \simeq V'' \simeq 0$, we have $\tau_0^{-1} \simeq 0$, and $\omega = -ik_p^2(t)/3H$. This is typical of diffusion-damping soft modes for which the relaxation time $\tau_{\mathbf{k}} \to \infty$ when $k_p \to 0$. These modes lead to long-range correlations in general [8].

Since perturbations $\phi(\mathbf{x}, t_0)$ only consist of modes with physical wavelength larger than $1/k_c$ at t_0, the Fourier transform of $\phi(\mathbf{x}, t_0)$ is limited to these modes as

$$\phi(\mathbf{x}, t_0) = \int_{k_p(t_0) < k_c} d^3\mathbf{k}\,\hat{\phi}_{\mathbf{k}}e^{-i\omega t_0 - i\mathbf{k}\cdot\mathbf{x}}. \tag{14}$$

We are concerned with perturbations $\phi(\mathbf{x}, t_0)$ localized within the Hubble radius $1/H$ ($< 1/k_c$) and in this range, the phase factor $\exp(-i\mathbf{k}\cdot\mathbf{x})$ in Eq. (14) is almost a constant. Moreover $\phi(\mathbf{x}, t_0)$ should be described by an ensemble of solutions of Eq. (8) with various realizations of η. Because η is white noise, the amplitude of fluctuations $\hat{\phi}_{\mathbf{k}}$ will be k-independent on average, i.e., $\langle \hat{\phi}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}^* \rangle \simeq \langle \hat{\phi}^2 \rangle$ is constant, where $\langle ... \rangle$ denotes averaging with respect to the ensemble of η. Hence, we have $\langle \phi\phi^* \rangle \equiv \langle \phi(\mathbf{x}, t_0)\phi^*(\mathbf{x}, t_0) \rangle \simeq k_c^6 \langle \hat{\phi}^2 \rangle$. The values $\langle \hat{\phi}^2 \rangle$ for quantum and thermal fluctuations have been calculated in [9,10].

The Fourier transform of $\phi(\mathbf{x}, t)$ is given by

$$\phi(\mathbf{x}, t) = \int_{k_p(t) < k_c} d^3\mathbf{k}\,\hat{\phi}_{\mathbf{k}}e^{-i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$
$$= \int_{k_p(t) < k_c} d^3\mathbf{k}\,\hat{\phi}_{\mathbf{k}}e^{-i\omega t_0 - i\mathbf{k}\cdot\mathbf{x}}\exp\left(-\left[\frac{1}{\tau_0} + \frac{k_p^2(t)}{3H}\right](t - t_0)\right). \tag{15}$$

Thus, the time-dependent correlation function is

$$\langle \phi(\mathbf{x}, t)\phi^*(\mathbf{x}, t_0) \rangle = k_c^3 \langle \hat{\phi}^2 \rangle \int_{k_p(t) < k_c} d^3\mathbf{k}\exp\left(-\left[\frac{1}{\tau_0} + \frac{k_p^2(t)}{3H}\right](t - t_0)\right). \tag{16}$$

In the case of a flat potential $\tau_0^{-1} = 0$, we have the long-time tail correlation function as

$$\frac{\langle \phi(\mathbf{x}, t)\phi^*(\mathbf{x}, t_0) \rangle}{\langle \phi\phi^* \rangle} \simeq \left(\frac{t_f}{t - t_0}\right)^{3/2} \quad \text{with} \quad (t - t_0) \gg t_f, \tag{17}$$

and

$$t_f = \frac{3H}{k_c^2} > \frac{1}{H}. \tag{18}$$

Comparing Eq. (17) with Eq. (4), we have as expected $d = 3$, and the "mean-free time" is determined by the cut-off $1/k_c$, i.e., the minimum scale of the fluctuations ϕ.

When the potential V is not constant, the time correlation function $\langle \phi(t)\phi(t_0) \rangle$ [Eq. (17)] contains an exponential decay factor $\exp[-(t - t_0)/\tau_0]$. However, the slow-roll dynamics requires that $V'' \ll H^2$. Thus, the factor $\exp[-(t - t_0)/\tau_0]$ can always be ignored as it is equal to about 1 provided $H(t - t_0)$ is not very large. Therefore, the long temporal correlations do not depend on the details of inflation. In fact, the slow-roll condition in inflation plays the same role as the critical slowing down in phase transitions. Under these conditions, the relaxation is dominated by soft modes and long range temporal correlations occur [8]. Because the background space is uniform, these soft modes do not induce long range spatial correlations and therefore, the perturbations in different space $\mathbf{x}$ are uncorrelated. In other words, the perturbation field has a Gaussian one-point distribution with respect to $\mathbf{x}$.

For inflation with thermal dissipation, Eq. (17) still holds [11]. In this case $1/k_c$ can be as small as the Hubble size $1/H$ [10], and therefore

$$t_f \simeq \frac{1}{H}, \tag{19}$$

i.e. the mean-free time is given by the Hubble time.

It is important to stress that although the long temporal correlations are calculated here using the stochastic formalism, this result is far more general than any specific stochastic inflation ansatz. The Langevin equation (10) for the long wavelength fluctuations of the scalar field can be derived from first principles by considering the quantum evolution of the reduced density matrix. Since it is necessary that quantum coherence be lost in order to treat the initial density perturbations classically with general relativity, a quantum-to-classical decoherence is necessary. The decoherence via a coarse-grained quantum temporal evolution produces classical correlations in phase space like Eq. (17) through the interaction between long wavelength modes and the short wavelength "thermal" noise of the Hawking temperature H [12]. Thus long temporal correlations are a generic feature of inflationary models with a slow-roll regime.

Since the evolution of perturbations from horizon-crossing to the time of decoupling is linear, we have

$$\tilde{\epsilon}_{k,\mathbf{x}} \propto \phi(\mathbf{x}, t_c), \tag{20}$$

where $\tilde{\epsilon}_{k,\mathbf{x}}$ is the wavelet coefficient of density or temperature fluctuations on scale k at position $\mathbf{x}$, and k is related to t_c via Eq. (3). Thus, the scale-scale correlations of the cosmic temperature fluctuations should be

$$\langle \tilde{\epsilon}_{k_1,x_1} \tilde{\epsilon}_{k_2,x_2} \rangle \propto \langle \phi(t_{c1}) \phi^*(t_{c2}) \rangle. \tag{21}$$

The scale-scale correlations of the cosmic temperature fluctuations have been detected using [2]

$$C(k_1, k_2, x) \equiv \frac{\langle \tilde{\epsilon}_{k_1,x}^2 \tilde{\epsilon}_{k_2,x}^2 \rangle}{\langle \tilde{\epsilon}_{k_1,x}^2 \rangle \langle \tilde{\epsilon}_{k_2,x}^2 \rangle}. \tag{22}$$

where the correlation is between wavelet coefficients at different scales but at the same physical location. From Eqs. (21) and (22), we have approximately that

$$C(k_1, k_2, x) - 1 \sim \left[\frac{\langle \phi(t_{c1}) \phi^*(t_{c2}) \rangle}{\langle \phi^2 \rangle} \right]^2. \tag{23}$$

Since $k_c \leq H$, the scale-scale correlation given by Eq. (17) is very significant. From Eq. (3), the time difference of forming perturbations on scales k and $2k$ is $\Delta Ht = \log 2$. Hence, $\langle \phi(t_{c1} + \log 2/H)\phi^*(t_{c1})\rangle/\langle \phi^2 \rangle$ is comparable to 1 according to Eq. (17). Therefore $C(k, 2k, x) - 1$ can also be as large as about 1 implying that long-time tails can produce scale-scale correlations on the same order as the temperature fluctuations.

The correlation (17) is localized, i.e., the fluctuations $\phi(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t_0)$ are at the same position $\mathbf{x}$ with size about $1/H^3$. Therefore,

$$\langle \phi(\mathbf{x}, t) \phi^*(\mathbf{x}', t_0) \rangle \simeq 0, \tag{24}$$

if $\mathbf{x}$ and $\mathbf{x}'$ are different. As a consequence, one can predict that the scale-scale correlations at different positions will be zero, or

$$C(k_1, k_2, x_1 - x_2) \equiv \frac{\langle \tilde{\epsilon}_{k_1,x_1}^2 \tilde{\epsilon}_{k_2,x_2}^2 \rangle}{\langle \tilde{\epsilon}_{k_1,x_1}^2 \rangle \langle \tilde{\epsilon}_{k_2,x_2}^2 \rangle} \sim 1, \tag{25}$$

where $x_1 - x_2 \neq 0$. We have tested this prediction for the Ly-α forests. The results show a good agreement with Eq. (25) [13].

It has generally been held that mode-mode coupling effects are negligible when perturbations are small and the linear approximation holds. The justification for this view can be seen from the decay of perturbations described by the linearized Boltzmann equation, which without considering diffusion effects, always decay exponentially. However this result is no longer true for processes in which diffusion-like long wavelength modes are important. In this case, even though the short wavelength modes decay exponentially [Eq. (9)], the long wavelength modes occur during the long-time tail which leads to scale-scale correlations [Eq. (17)]. Furthermore scale-scale correlations also arise in the conventional approach to inflation due to the loss of quantum coherence via the mechanism of coarse-grained histories [12]. As such, scale-scale correlations are a generic feature of the inflation scenario with a slow-roll regime. The detected $C(11°, 22°) \sim 2$ [2] implies that first evidence of inflation induced scale-scale correlations may have been found. Finally, the detection of scale-scale correlations provides a way to reveal the time correlations of the fluctuations during inflation.

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